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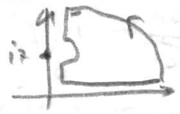
(1)

$$\int_0^\infty \frac{x^2\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{\frac32}} J_1(ax)\,dx = -\frac{\partial}{\partial a}\int_0^\infty \frac{x\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{\frac32}}J_0(ax)\,dx = -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^\infty\int_0^\infty \frac{x\sin(ax\,\mathrm{ch}\,t)\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{\frac32}}\,dt\,dx$$

$$\int_0^\infty \frac{x\sin ax \sin\sqrt{x^2+z^2}}{(x^2+z^2)^{3/2}}dx = (a<1)\ \frac{1}{2i}\int_0^\infty \frac{x\sin ax\cdot e^{i\sqrt{x^2+z^2}}}{(x^2+z^2)^{\frac32}}dx + \text{h.c.} =$$



$$= \frac{1}{2i}\lim_{r\to0}\left(\int_0^{z-r}\frac{iy\cdot i\,\mathrm{sh}\,ay\cdot e^{i\sqrt{z^2-y^2}}}{(z^2-y^2)^{3/2}}\, i\,dy + \int_{z+r}^\infty \frac{iy\cdot i\,\mathrm{sh}\,ay\cdot e^{-\sqrt{y^2-z^2}}}{-i(y^2-z^2)^{3/2}}\, i\,dy + \int_{-\pi/2}^{\pi/2}\frac{(iz+re^{i\varphi})\sin(a(iz+re^{i\varphi}))e^{i\sqrt{re^{i\varphi}(re^{i\varphi}+2iz)}}}{(re^{i\varphi}(re^{i\varphi}+2iz))^{\frac32}}\, ire^{i\varphi}d\varphi\right) + \text{h.c.}$$

$$= \lim_{r\to0}\left(-\int_0^{z-r}\frac{y\,\mathrm{sh}\,ay\cos\sqrt{z^2-y^2}}{(z^2-y^2)^{3/2}}dy + \frac{z}{2\sqrt r}\left[\int_{-\pi/2}^{\pi/2} e^{-i\frac{\varphi}{2}}\frac{\sin(a(iz+re^{i\varphi}))\cdot e^{i\sqrt{re^{i\varphi}(re^{i\varphi}+2iz)}}}{(re^{i\varphi}+2iz)^{\frac32}}\, i\,d\varphi + \underbrace{\int_{-\pi/2}^{\pi/2} e^{i\frac{\varphi}{2}}\frac{\sin(a(-iz+re^{-i\varphi}))e^{-i\sqrt{re^{-i\varphi}(re^{-i\varphi}-2iz)}}}{(re^{-i\varphi}-2iz)^{\frac32}}(-i)\,d\varphi}_{\varphi\to-\varphi}\right]\right) =$$

$$\frac{\partial^2}{\partial z\partial y}\sin\sqrt{z^2-y^2} = \frac{\partial}{\partial z}\left(-\frac{y\cos\sqrt{z^2-y^2}}{\sqrt{z^2-y^2}}\right) = \frac{yz\sin\sqrt{z^2-y^2}}{z^2-y^2} + \frac{yz\cos\sqrt{z^2-y^2}}{(z^2-y^2)^{\frac32}}$$

$$= \lim_{r\to0}\left[-\int_0^{z-r}\mathrm{sh}\,ay\left(\frac1z\frac{\partial^2}{\partial z\partial y}\sin\sqrt{z^2-y^2} - \frac{y\sin\sqrt{z^2-y^2}}{z^2-y^2}\right)dy + \frac{iz}{2\sqrt r}\int_{-\pi/2}^{\pi/2}e^{-i\frac{\varphi}{2}}\left(\frac{i\,\mathrm{sh}\,az\left(1+i\sqrt{re^{i\varphi}(re^{i\varphi}+2iz)}\right)}{(2iz)^{3/2}} - \frac{i\,\mathrm{sh}(-az)\left(1-i\sqrt{re^{i\varphi}(re^{i\varphi}-2iz)}\right)}{(-2iz)^{3/2}}\right)d\varphi\right]$$

$$= \lim_{r\to0}\left[-\frac{\mathrm{sh}\,ay}{z}\frac{\partial}{\partial z}\sin\sqrt{z^2-y^2}\Big|_0^{z-r} + \int_0^{z-r}\frac{a\,\mathrm{ch}\,ay}{z}\frac{\partial}{\partial z}\sin\sqrt{z^2-y^2}\,dy + \int_0^{z-r}\frac{y\,\mathrm{sh}\,ay\sin\sqrt{z^2-y^2}}{z^2-y^2}dy + \frac{iz}{2\sqrt r}\cdot\frac{i\,\mathrm{sh}\,az}{(2z)^{3/2}}\int_{-\pi/2}^{\pi/2}e^{-i\frac{\varphi}{2}}\left(i^{-\frac32}+(-i)^{-\frac32}+\sqrt{re^{i\varphi}}\left(\sqrt{2z}+\sqrt{2z}\right)\right)d\varphi\right]$$

$$= \lim_{r\to0}\left[-\mathrm{sh}\,ay\cdot\frac{\cos\sqrt{z^2-y^2}}{\sqrt{z^2-y^2}}\Big|_0^{z-r} + a\int_0^{z-r}\frac{\mathrm{ch}\,ay\cos\sqrt{z^2-y^2}}{\sqrt{z^2-y^2}}dy + \int_0^{z-r}\frac{y\,\mathrm{sh}\,ay\sin\sqrt{z^2-y^2}}{z^2-y^2}dy - \frac{\mathrm{sh}\,az}{4\sqrt{2rz}}\left(\left(e^{-\frac{3\pi i}{4}}+e^{\frac{3\pi i}{4}}\right)\cdot\left(-\frac2i e^{-i\frac{\varphi}{2}}\right)\Big|_{-\frac\pi2}^{\frac\pi2} + \sqrt r\cdot2\sqrt{2z}\cdot\pi\right)\right]$$

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$$= a\cdot\frac\pi2 J_0\left(z\sqrt{1-a^2}\right) + \int_0^z\frac{y\,\mathrm{sh}\,ay\sin\sqrt{z^2-y^2}}{z^2-y^2}dy + \lim_{r\to0}\left[-\frac{\mathrm{sh}(a(z-r))\cos\sqrt{2rz}}{\sqrt{2rz}} - \frac{\mathrm{sh}\,az}{4\sqrt{2rz}}\left(-\sqrt2\right)\left(-\frac2i\right)\left(e^{-\frac{\pi i}{4}}-e^{\frac{\pi i}{4}}\right)+2\pi\sqrt{2rz}\right]$$

$$= \frac{\pi a}{2}J_0\left(z\sqrt{1-a^2}\right) + \int_0^z \frac{y\,\mathrm{sh}\,ay\sin\sqrt{z^2-y^2}}{z^2-y^2}dy - \frac\pi2\,\mathrm{sh}\,az + \lim_{r\to0}\left(-\frac{\mathrm{sh}\,az}{\sqrt{2rz}}\left(1+\frac14\cdot\frac{2\sqrt2}{i}\cdot\left(-i\sqrt2\right)\right)\right) =$$

$$= -\frac\pi2\,\mathrm{sh}\,az + \frac{\pi a}{2}J_0\left(z\sqrt{1-a^2}\right) + \int_0^z\frac{y\,\mathrm{sh}\,ay\sin\sqrt{z^2-y^2}}{z^2-y^2}dy \xrightarrow[a\to1]{} -\frac\pi2\,\mathrm{sh}\,z+\frac\pi2+\frac\pi2(\mathrm{ch}\,z-1) = \frac\pi2 e^{-z}$$

$$\int_0^\infty\frac{x\sin ax\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{\frac32}}dx = (a>1) = \frac{1}{2i}\int_0^\infty\frac{xe^{iax}\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{3/2}}dx+\text{h.c.} =$$

$$= \frac1{2i}\lim_{r\to0}\left(\int_0^{z-r}\frac{iye^{-ay}\sin\sqrt{z^2-y^2}}{(z^2-y^2)^{3/2}}\, i\,dy + \int_{z+r}^\infty\frac{iye^{-ay}\cdot i\,\mathrm{sh}\sqrt{y^2-z^2}}{-i(y^2-z^2)^{3/2}}\, i\,dy + \int_{-\pi/2}^{\pi/2}\frac{(iz+re^{i\varphi})e^{ia(iz+re^{i\varphi})}\sin\sqrt{re^{i\varphi}(re^{i\varphi}+2iz)}}{(re^{i\varphi}(re^{i\varphi}+2iz))^{\frac32}}\, ire^{i\varphi}d\varphi + \text{h.c.}\right) =$$

$$\frac1{2i}\cdot\frac{ize^{-az}}{2iz}\cdot\pi i + \text{h.c.} = \frac\pi2 e^{-az}\xrightarrow[a\to1]{}\frac\pi2e^{-z}$$

$$\int_0^\infty\frac{x^2\sin\sqrt{x^2+z^2}}{(x^2+z^2)^{3/2}}J_1(ax)\,dx = (a>1) = -\frac2\pi\frac{\partial}{\partial a}\int_0^\infty\frac\pi2e^{-az\,\mathrm{ch}\,t}dt = -\frac{\partial}{\partial a}K_0(az) = zK_1(az)$$

$$= (a<1) = -\frac2\pi\frac{\partial}{\partial a}\left[\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\left(-\frac\pi2\,\mathrm{sh}(az\,\mathrm{ch}\,t)+\frac\pi2a\,\mathrm{ch}\,t\,J_0\left(z\sqrt{1-a^2\mathrm{ch}^2t}\right) + \int_0^z\frac{y\,\mathrm{sh}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}dy\right)dt + \int_{\ln\frac{1+\sqrt{1-a^2}}{a}}^\infty\frac\pi2e^{-az\,\mathrm{ch}\,t}dt\right]$$

$$= -\frac{\partial}{\partial a}\left[\int_0^\infty e^{-az\,\mathrm{ch}\,t}dt + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\left(-\mathrm{ch}(az\,\mathrm{ch}\,t)+\frac2\pi\int_0^z\frac{y\,\mathrm{sh}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}dy\right)dt + a\cdot\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}\right] =$$

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$$= -\frac{\partial}{\partial a} K_0(az) - \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(-z\,\mathrm{sh}(az\,\mathrm{ch}\,t) + \frac{2}{\pi} \int_0^z \frac{y^2\,\mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy \right) \mathrm{ch}\,t\,dt - \frac{a\cos(z\sqrt{1-a^2})}{\sqrt{1-a^2}} \overset{5}{+} \mathrm{ch}\,z - \frac{2}{\pi}\cdot\frac{\pi}{2}(\mathrm{ch}\,z - 1) =$$

$$= z K_1(az) + 1 - \frac{a\cos(z\sqrt{1-a^2})}{\sqrt{1-a^2}} + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(z\,\mathrm{sh}(az\,\mathrm{ch}\,t) - \frac{2}{\pi}\left[-\int_0^z \mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}\,dy + z^2 \int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy \right] \right) \mathrm{ch}\,t\,dt =$$

$$= z K_1(az) + 1 - \frac{a\cos(z\sqrt{1-a^2})}{\sqrt{1-a^2}} + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(z\,\mathrm{sh}(az\,\mathrm{ch}\,t) - \frac{2z^2}{\pi} \int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy - \frac{2}{\pi}\,\overset{8}{\frac{\pi z J_1(z\sqrt{1-a^2\mathrm{ch}^2 t})}{2\sqrt{1-a^2\mathrm{ch}^2 t}}} \right) \mathrm{ch}\,t\,dt =$$

$$= z K_1(az) + 1 - \frac{a\cos(z\sqrt{1-a^2})}{\sqrt{1-a^2}} + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(z\,\mathrm{sh}(az\,\mathrm{ch}\,t) - \frac{2z^2}{\pi} \int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy \right) \mathrm{ch}\,t\,dt + z\,\frac{\cos(z\sqrt{1-a^2})-1}{az\sqrt{1-a^2}} \overset{7}{=}$$

$$= z K_1(az) + 1 - \frac{1}{a\sqrt{1-a^2}} + \frac{\cos(z\sqrt{1-a^2})}{\sqrt{1-a^2}}\cdot\frac{1-a^2}{a} + \ldots =$$

$$= z K_1(az) + 1 - \frac{1}{a\sqrt{1-a^2}} + \frac{\sqrt{1-a^2}}{a}\cos(z\sqrt{1-a^2}) + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(z\,\mathrm{sh}(az\,\mathrm{ch}\,t) - \frac{2z^2}{\pi} \int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}\,t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy \right) \mathrm{ch}\,t\,dt =$$

$$= z K_1(az) + 1 - \frac{1}{a\sqrt{1-a^2}} + \frac{\sqrt{1-a^2}}{a}\cos(z\sqrt{1-a^2}) + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} z\,\mathrm{sh}(az\,\mathrm{ch}\,t) - \frac{2z^2}{\pi}\left(\frac{\pi}{2z}\,\mathrm{sh}(az\,\mathrm{ch}\,t) + \frac{\pi}{z}\sum_{n=0}^{\infty} \left(\frac{1-a\,\mathrm{ch}\,t}{1+a\,\mathrm{ch}\,t}\right)^{\frac{2n+1}{2}} \overset{9}{J_{2n+1}}\left(z\sqrt{1-a^2\mathrm{ch}^2 t}\right) \right) \mathrm{ch}\,t\,dt$$

$$= z K_1(az) + 1 - \frac{1}{a\sqrt{1-a^2}} + \frac{\sqrt{1-a^2}}{a}\cos(z\sqrt{1-a^2}) - 2z \sum_{n=0}^{\infty} \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} \left(\frac{1-a\,\mathrm{ch}\,t}{1+a\,\mathrm{ch}\,t}\right)^{n+\frac{1}{2}} J_{2n+1}\left(z\sqrt{1-a^2\mathrm{ch}^2 t}\right) \mathrm{ch}\,t\,dt$$

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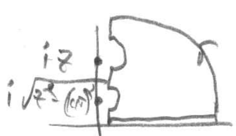
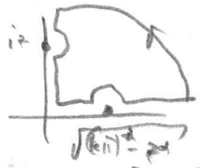
$$\int_0^\infty \frac{x^2\sin\sqrt{x^2+z^2}\,J_1(ax)}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx = -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^\infty\int_0^\infty \frac{x\sin(ax\,\mathrm{ch}\,t)\sin\sqrt{x^2+z^2}}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx\,dt$$



$z>k\pi$



$z<k\pi$

$$\int_0^\infty \frac{x\sin(ax)\sin\sqrt{x^2+z^2}}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx \overset{5}{=} (a<1) = \frac{1}{2i}\int_0^\infty \frac{x\sin ax\, e^{i\sqrt{x^2+z^2}}}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx + \text{h.c.} =$$

$$=(z>k\pi) = \frac{1}{2i}\Big[\int_0^z \frac{iy\, i\,\mathrm{sh}\,ay\, e^{i\sqrt{z^2-y^2}}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}\, i\,dy + \frac{i\sqrt{z^2-(k\pi)^2}\, i\,\mathrm{sh}\sqrt{z^2-(k\pi)^2}\,e^{ik\pi}}{2i\sqrt{z^2-(k\pi)^2}\cdot k\pi}\cdot\pi i + \int_z^\infty \frac{iy\cdot i\,\mathrm{sh}\,ay\cdot e^{-\sqrt{y^2-z^2}}}{-i(y^2+(k\pi)^2-z^2)\sqrt{y^2-z^2}}\, i\,dy\Big] + \text{h.c.}$$

$$= -\int_0^z \frac{y\,\mathrm{sh}\,ay\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy \xrightarrow[a\to1]{} 0 \quad 5$$

$$\overset{5}{=}(a<1,\ z<k\pi) = -\int_0^z \frac{y\,\mathrm{sh}\,ay\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy + \frac{\sqrt{(k\pi)^2-z^2}\,\sin\left(a\sqrt{(k\pi)^2-z^2}\right)e^{ik\pi}}{2\sqrt{(k\pi)^2-z^2}\cdot k\pi}\cdot(-\pi i)\cdot\frac{1}{2i} =$$

$$= -\int_0^z \frac{y\,\mathrm{sh}\,ay\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}} + \frac{(-1)^{k+1}}{4k}\sin\left(a\sqrt{(k\pi)^2-z^2}\right) \xrightarrow[a\to1]{} -\underset{5}{\frac{(-1)^{k+1}}{4k}\sin\sqrt{(k\pi)^2-z^2}} + \frac{(-1)^{k+1}}{4k}\sin\sqrt{(k\pi)^2-z^2} = 0$$

$$\overset{5}{=}(a>1) = \frac{1}{2i}\int_0^\infty \frac{x e^{iax}\sin\sqrt{x^2+z^2}}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx + \text{h.c.} =$$

$$=(z>k\pi) = \frac{1}{2i}\Big[\int_0^z \frac{iy\, e^{-ay}\sin\sqrt{z^2-y^2}}{(z^2-(k\pi)^2-y^2)\sqrt{z^2-y^2}}\, i\,dy + \int_z^\infty \frac{iy\, e^{-ay}\, i\,\mathrm{sh}\sqrt{y^2-z^2}}{-i(y^2-z^2+(k\pi)^2)\sqrt{y^2-z^2}}\, i\,dy + \frac{i\sqrt{z^2-(k\pi)^2}\cdot e^{-a\sqrt{z^2-(k\pi)^2}}\sin k\pi}{2i\sqrt{z^2-(k\pi)^2}\cdot k\pi}\cdot\pi i\Big] + \text{h.c.} = 0$$

$$=(a>1,\ z<k\pi) = \frac{\sqrt{(k\pi)^2-z^2}\, e^{ia\sqrt{(k\pi)^2-z^2}}\sin k\pi}{2\sqrt{(k\pi)^2-z^2}\; k\pi}\cdot(-\pi i)\cdot\frac{1}{2i} + \text{h.c.} = 0$$

$$\int_0^\infty \frac{x^2\sin\sqrt{x^2+z^2}\,J_1(ax)}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx = (a<1,\ z>k\pi) = -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} -\int_0^z \frac{y\,\mathrm{sh}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy\,dt =$$

$$= \frac{2}{\pi}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\int_0^z \frac{y^2\,\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}\,\mathrm{ch}t\,dt + \frac{2}{\pi}\int_0^z \frac{y\,\mathrm{sh}\,y\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy = \frac{2}{\pi}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Big(-\int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{\sqrt{z^2-y^2}}dy +$$

$$+(z^2-(k\pi)^2)\int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy\Big)\mathrm{ch}t\,dt = \frac{2}{\pi}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Big(-\underset{5}{\frac{\pi}{2}J_0\left(z\sqrt{1-a^2\mathrm{ch}^2t}\right)} + (z^2-(k\pi)^2)\int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy\Big)\mathrm{ch}t\,dt =$$

$$= -\frac{\sin(z\sqrt{1-a^2})}{az} + \frac{2(z^2-(k\pi)^2)}{\pi}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy\,\mathrm{ch}t\,dt$$

$$\int_0^\infty \frac{x^2\sin\sqrt{x^2+z^2}\,J_1(ax)}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx = (a<1,\ z<k\pi) = -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Big(-\int_0^z \frac{y\,\mathrm{sh}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy + \frac{(-1)^{k+1}}{4k}\sin\left(a\,\mathrm{ch}t\sqrt{(k\pi)^2-z^2}\right)\Big)dt =$$

$$= -\frac{\sin(z\sqrt{1-a^2})}{az} + \int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Big(\frac{2(z^2-(k\pi)^2)}{\pi}\int_0^z \frac{\mathrm{ch}(ay\,\mathrm{ch}t)\cos\sqrt{z^2-y^2}}{(z^2-y^2-(k\pi)^2)\sqrt{z^2-y^2}}dy + \frac{(-1)^k}{2k\pi}\sqrt{(k\pi)^2-z^2}\cos\left(a\,\mathrm{ch}t\sqrt{(k\pi)^2-z^2}\right)\Big)\mathrm{ch}t\,dt$$

$$\int_0^\infty \frac{x^2\sin\sqrt{x^2+z^2}\,J_1(ax)}{(x^2+z^2-(k\pi)^2)\sqrt{x^2+z^2}}dx = (a>1) = 0$$

$$\int_0^\infty \frac{x^2 \sin\sqrt{x^2+z^2}\, J_1(ax)}{\left(x^2+z^2-(k\pi)^2\right)\left(x^2+z^2\right)^{\frac{3}{2}}}\,dx = \frac{1}{(k\pi)^2}\int_0^\infty\left(\frac{1}{x^2+z^2-(k\pi)^2}-\frac{1}{x^2+z^2}\right)\frac{x^2\sin\sqrt{x^2+z^2}}{\sqrt{x^2+z^2}}J_1(ax)\,dx =$$

$$(a>1) = -\frac{zK_1(az)}{(k\pi)^2}$$

$$(a<1,\ z>k\pi) = \frac{1}{(k\pi)^2}\Bigg[-\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}+\frac{2\left(z^2-(k\pi)^2\right)}{\pi}\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\int_0^z \frac{\operatorname{ch}(ay\operatorname{ch}t)\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dy\,\operatorname{ch}t\,dt - zK_1(az)-1+\frac{1}{a\sqrt{1-a^2}} -$$

$$-\frac{\sqrt{1-a^2}}{a}\cos\left(z\sqrt{1-a^2}\right)-\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\left(z\operatorname{sh}(az\operatorname{ch}t)-\frac{2z^2}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy\right)\operatorname{ch}t\,dt\Bigg] =$$

$$= \frac{1}{(k\pi)^2}\Bigg(-zK_1(az)-1+\frac{1}{a\sqrt{1-a^2}}-\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}-\frac{\sqrt{1-a^2}}{a}\cos\left(z\sqrt{1-a^2}\right)+\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Bigg[\frac{2\left(z^2-(k\pi)^2\right)}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dy +$$

$$+\frac{2z^2}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy - z\operatorname{sh}(az\operatorname{ch}t)\Bigg]\operatorname{ch}t\,dt\Bigg) =$$

$$= \frac{1}{(k\pi)^2}\Bigg(-zK_1(az)-1+\frac{1}{a\sqrt{1-a^2}}-\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}-\frac{\sqrt{1-a^2}}{a}\cos\left(z\sqrt{1-a^2}\right)+\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Bigg\{\frac{2\left(z^2-(k\pi)^2\right)}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dt +$$

$$+2z\sum_{n=0}^\infty\left(\frac{1-a\operatorname{ch}t}{1+a\operatorname{ch}t}\right)^{n+\frac{1}{2}}J_{2n+1}\left(z\sqrt{1-a^2\operatorname{ch}^2t}\right)\Bigg]\operatorname{ch}t\,dt\Bigg)$$

$$(a<1,\ z<k\pi) = \frac{1}{(k\pi)^2}\Bigg(-zK_1(az)-1+\frac{1}{a\sqrt{1-a^2}}-\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}-\frac{\sqrt{1-a^2}}{a}\cos\left(z\sqrt{1-a^2}\right)+\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Bigg[\frac{2\left(z^2-(k\pi)^2\right)}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dy +$$

$$+\frac{(-1)^k}{2k\pi}\sqrt{(k\pi)^2-z^2}\cos\left(a\operatorname{ch}t\sqrt{(k\pi)^2-z^2}\right)+\frac{2z^2}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy - z\operatorname{sh}(az\operatorname{ch}t)\Bigg]\operatorname{ch}t\Bigg)$$

$$\int_0^\infty \frac{x^2 \sin\sqrt{x^2+z^2}\, J_1(ax)}{\left(x^2+z^2-(k\pi)^2\right)\left(x^2+z^2\right)^{\frac{3}{2}}}\,dx = \frac{1}{(k\pi)^2}\Bigg(-zK_1(az)+\Theta(1-a)\cdot\Bigg\{-1+\frac{1}{a\sqrt{1-a^2}}-\frac{\sin\left(z\sqrt{1-a^2}\right)}{az}-\frac{\sqrt{1-a^2}}{a}\cos\left(z\sqrt{1-a^2}\right)+$$

$$+\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\Bigg[\frac{2\left(z^2-(k\pi)^2\right)}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dy+\Theta(k\pi-z)\cdot\frac{(-1)^k}{2k\pi}\sqrt{(k\pi)^2-z^2}\cos\left(a\operatorname{ch}t\sqrt{(k\pi)^2-z^2}\right)+$$

$$+\underbrace{\frac{2z^2}{\pi}\int_0^z\frac{\operatorname{ch}(ay\operatorname{ch}t)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy - z\operatorname{sh}(az\operatorname{ch}t)}_{2z\sum_{n=0}^\infty\left(\frac{1-a\operatorname{ch}t}{1+a\operatorname{ch}t}\right)^{n+\frac{1}{2}}J_{2n+1}\left(z\sqrt{1-a^2\operatorname{ch}^2t}\right)}\Bigg]\operatorname{ch}t\,dt\Bigg\}\Bigg)$$

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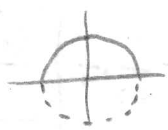
$$\int_0^z \frac{\operatorname{ch} ay \cos\sqrt{z^2-y^2}}{\sqrt{z^2-y^2}}\,dy = |y=z\cos\theta| = \int_{\pi/2}^0 \frac{\operatorname{ch}(az\cos\theta)\cos(z\sin\theta)}{z\sin\theta}\cdot(-z\sin\theta)\,d\theta = \frac12\int_0^\pi \operatorname{ch}(az\cos\theta)\cos(z\sin\theta)\,d\theta = |x=e^{i\theta}| =$$ 5

$$=\frac12\int \frac12\left(e^{\frac{az}{2}\left(x+\frac1x\right)}+e^{-\frac{az}{2}\left(x+\frac1x\right)}\right)\cdot\frac12\left(e^{\frac z2\left(x-\frac1x\right)}+e^{-\frac z2\left(x-\frac1x\right)}\right)\frac{dx}{ix} = \frac{1}{8i}\int\left(e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)}+e^{\frac z2\left((a-1)x+\frac{a+1}{x}\right)}+e^{-\frac z2\left((a+1)x+\frac{a-1}{x}\right)}+e^{-\frac z2\left((a-1)x+\frac{a+1}{x}\right)}\right)\frac{dx}{x} ⊜$$



$$\int e^{\frac z2\left((a-1)x+\frac{a+1}{x}\right)}\frac{dx}{x} = \left|x=\frac1u\right| = \int e^{\frac z2\left(\frac{a-1}{u}+(a+1)u\right)}\cdot u\cdot\left(-\frac{1}{u^2}\right)du = -\int e^{\frac z2\left((a+1)u+\frac{a-1}{u}\right)}\frac{du}{u}$$

$$⊜ \frac{1}{8i}\oint_{|x|=1}\left(e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)}+e^{-\frac z2\left((a+1)x+\frac{a-1}{x}\right)}\right)\cdot\frac{dx}{x} ⊜$$

$$\frac1x e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)} = x^{-1}\sum_{n=0}^\infty\left(\frac{a+1}{2}\cdot zx\right)^n\cdot\frac{1}{n!}\sum_{k=0}^\infty\left(\frac{a-1}{2x}z\right)^k\cdot\frac{1}{k!} = \sum_{n=0}^\infty\sum_{k=0}^\infty\frac{(a+1)^n(a-1)^k}{n!\,k!}\left(\frac z2\right)^{n+k}\cdot x^{n-k-1};\quad n-k-1=-1\to n=k$$

$$\oint_{|x|=1} e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)}\frac{dx}{x} = 2\pi i\cdot\sum_{n=0}^\infty\frac{(a+1)^n(a-1)^n}{(n!)^2}\left(\frac z2\right)^{2n} = 2\pi i\cdot\sum_{n=0}^\infty\frac{1}{(n!)^2}\left(-\frac{z^2}{4}(1-a^2)\right)^n = 2\pi i\cdot J_0\left(z\sqrt{1-a^2}\right)$$

$$⊜ \frac{1}{8i}\cdot 2\pi i\left(J_0\left(z\sqrt{1-a^2}\right)+J_0\left(-z\sqrt{1-a^2}\right)\right) = \frac\pi2 J_0\left(z\sqrt{1-a^2}\right)$$

$$\int_0^z\frac{y\operatorname{sh}y\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy = |y=z\cos\theta| = \int_{\pi/2}^0\frac{z\cos\theta\operatorname{sh}(z\cos\theta)\sin(z\sin\theta)}{z^2\sin^2\theta}(-z\sin\theta)\,d\theta = \frac12\int_0^\pi\frac{\operatorname{sh}(z\cos\theta)\sin(z\sin\theta)}{\operatorname{tg}\theta}\,d\theta = |x=e^{i\theta}| =$$

$$=\frac12\int\frac12\left(e^{\frac z2\left(x+\frac1x\right)}-e^{-\frac z2\left(x+\frac1x\right)}\right)\cdot\frac{1}{2i}\left(e^{\frac z2\left(x-\frac1x\right)}-e^{-\frac z2\left(x-\frac1x\right)}\right)\cdot\frac{\frac12\left(x+\frac1x\right)}{\frac{1}{2i}\left(x-\frac1x\right)}\frac{dx}{ix} = \frac{1}{8i}\int\left(e^{zx}-e^{-\frac zx}-e^{\frac zx}+e^{-zx}\right)\frac{x+\frac1x}{x-\frac1x}\frac{dx}{x} = \frac{1}{8i}\oint_{|x|=1}\left(e^{zx}+e^{-zx}\right)\frac{x+\frac1x}{x-\frac1x}\frac{dx}{x}$$

$$=\frac{1}{4i}\oint_{|x|=1}\frac{(x^2+1)\operatorname{ch}zx}{x(x^2-1)}\,dx = \frac{1}{4i}\left(2\pi i(-1)+\pi i\left(\frac{2\operatorname{ch}z}{2}+\frac{2\operatorname{ch}(-z)}{-(-2)}\right)\right) = -\frac\pi2+\frac\pi2\operatorname{ch}z = \frac\pi2(\operatorname{ch}z-1)$$

$$\int_0^z\frac{y\operatorname{sh}y\cos\sqrt{z^2-y^2}}{\left(z^2-y^2-(k\pi)^2\right)\sqrt{z^2-y^2}}\,dy = |y=z\cos\theta| = \int_{\pi/2}^0\frac{z\cos\theta\operatorname{sh}(z\cos\theta)\cos(z\sin\theta)}{\left(z^2\sin^2\theta-(k\pi)^2\right)z\sin\theta}(-z\sin\theta)\,d\theta = \frac z2\int_0^\pi\frac{\operatorname{sh}(z\cos\theta)\cos(z\sin\theta)\cos\theta}{z^2\sin^2\theta-(k\pi)^2}\,d\theta = |x=e^{i\theta}|$$

$$=\frac z2\int\frac{\frac12\left(e^{\frac z2\left(x+\frac1x\right)}-e^{-\frac z2\left(x+\frac1x\right)}\right)\cdot\frac12\left(e^{\frac z2\left(x-\frac1x\right)}+e^{-\frac z2\left(x-\frac1x\right)}\right)\cdot\frac12\left(x+\frac1x\right)}{-\frac{z^2}{4}\left(x-\frac1x\right)^2-(k\pi)^2}\frac{dx}{ix} = -\frac{z}{4i}\int\frac{\left(e^{zx}-e^{-\frac zx}+e^{\frac zx}-e^{-zx}\right)\left(x+\frac1x\right)}{z^2\left(x-\frac1x\right)^2+4(k\pi)^2}\frac{dx}{x} =$$

$$=-\frac{z}{4i}\oint_{|x|=1}\frac{\left(e^{zx}-e^{-zx}\right)\left(x+\frac1x\right)}{z^2\left(x-\frac1x\right)^2+4(k\pi)^2}\frac{dx}{x} = -\frac{z}{2i}\oint_{|x|=1}\frac{(x^2+1)\operatorname{sh}zx}{z^2(x^2-1)^2+(2k\pi x)^2}\,dx ⊜$$

$z>k\pi$

$z^2(x^2-1)^2+(2k\pi x)^2=0$

$z(x^2-1)=\pm 2ik\pi x$

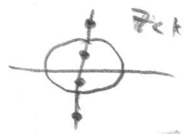
$z^2x^2\mp 2ik\pi x-z=0$

$$x=\frac{\pm ik\pi\pm'\sqrt{-(k\pi)^2+z^2}}{z}$$

$$|x|=\frac{(k\pi)^2+z^2-(k\pi)^2}{z^2}=1\ (z>k\pi)$$

$$x^2+1=\left(\frac{\pm\sqrt{z^2-(k\pi)^2}\pm' ik\pi}{z}\right)^2+1=\frac{z^2-(k\pi)^2-(k\pi)^2\pm' 2ik\pi\sqrt{z^2-k\pi^2}}{z^2}+1=\frac{2\sqrt{z^2-(k\pi)^2}}{z^2}\left(\sqrt{z^2-(k\pi)^2}\pm' ik\pi\right)$$

$z<k\pi$



$$\circleddash(z>k\pi) = -\frac{z}{2i}\cdot\frac{1}{z^2}\oint\limits_{|x|=1}\frac{(x^2+1)\,\mathrm{sh}\,zx}{\left(x-\frac{\sqrt{z^2-(k\pi)^2}-ik\pi}{z}\right)\left(x-\frac{\sqrt{z^2-(k\pi)^2}+ik\pi}{z}\right)\left(x+\frac{\sqrt{z^2-(k\pi)^2}+ik\pi}{z}\right)\left(x+\frac{\sqrt{z^2-(k\pi)^2}-ik\pi}{z}\right)}\,dx =$$

$$= -\frac{1}{2iz}\cdot\pi i\left[\frac{\frac{2\sqrt{z^2-(k\pi)^2}}{z^2}\left(\sqrt{z^2-(k\pi)^2}-ik\pi\right)\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}-ik\pi\right)}{-\frac{2ik\pi}{z}\,\frac{2\sqrt{z^2-(k\pi)^2}}{z}\,2\frac{\sqrt{z^2-(k\pi)^2}-ik\pi}{z}} + \frac{\frac{2\sqrt{z^2-(k\pi)^2}}{z^2}\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)}{\frac{2ik\pi}{z}\,2\frac{\sqrt{z^2-(k\pi)^2}+ik\pi}{z}\,\frac{2\sqrt{z^2-(k\pi)^2}}{z}}\right. +$$

$$\left. + \frac{\frac{2\sqrt{z^2-(k\pi)^2}}{z^2}\left(\sqrt{z^2-(k\pi)^2}-ik\pi\right)\mathrm{sh}\left(-\sqrt{z^2-(k\pi)^2}+ik\pi\right)}{-2\frac{\sqrt{z^2-(k\pi)^2}-ik\pi}{z}\left(-\frac{2\sqrt{z^2-(k\pi)^2}}{z}\right)\frac{2ik\pi}{z}} + \frac{\frac{2\sqrt{z^2-(k\pi)^2}}{z}\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)\mathrm{sh}\left(-\sqrt{z^2-(k\pi)^2}-ik\pi\right)}{-\frac{2\sqrt{z^2-(k\pi)^2}}{z}\left(-\frac{2\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)}{z}\right)\left(-\frac{2ik\pi}{z}\right)}\right] =$$

$$= \frac{\pi}{2z}\cdot\frac{z}{4ik\pi}\left(-\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}-ik\pi\right)+\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)+\mathrm{sh}\left(-\sqrt{z^2-(k\pi)^2}+ik\pi\right)-\mathrm{sh}\left(-\sqrt{z^2-(k\pi)^2}-ik\pi\right)\right) =$$

$$= \frac{1}{8ik}\cdot 2\left(\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}+ik\pi\right)-\mathrm{sh}\left(\sqrt{z^2-(k\pi)^2}-ik\pi\right)\right) = \frac{1}{4ik}\,\mathrm{ch}\sqrt{z^2-(k\pi)^2}\,\mathrm{sh}(ik\pi) = \frac{1}{4ik}\,\mathrm{ch}\sqrt{z^2-(k\pi)^2}\cdot i\sin k\pi = 0$$

$$\circleddash(z<k\pi) = -\frac{z}{2i}\cdot\frac{1}{z^2}\oint\limits_{|x|=1}\frac{(x^2+1)\,\mathrm{sh}\,zx}{\left(x-i\frac{\sqrt{(k\pi)^2-z^2}+k\pi}{z}\right)\left(x-i\frac{\sqrt{(k\pi)^2-z^2}-k\pi}{z}\right)\left(x+i\frac{\sqrt{(k\pi)^2-z^2}+k\pi}{z}\right)\left(x+i\frac{\sqrt{(k\pi)^2-z^2}-k\pi}{z}\right)} =$$

$$= -\frac{1}{2iz}\cdot 2\pi i\cdot\frac{2i\sqrt{(k\pi)^2-z^2}}{z^2}\cdot i\left(\sqrt{(k\pi)^2-z^2}-k\pi\right)\cdot\mathrm{sh}\left(i\left(\sqrt{(k\pi)^2-z^2}-k\pi\right)\right)\left[\frac{1}{-\frac{2ik\pi}{z}\cdot\frac{2i\sqrt{(k\pi)^2-z^2}}{z}\cdot 2i\frac{\sqrt{(k\pi)^2-z^2}-k\pi}{z}}\right. -$$

$$\left. - \frac{1}{-2i\frac{\sqrt{(k\pi)^2-z^2}-k\pi}{z}\left(-2i\frac{\sqrt{(k\pi)^2-z^2}}{z}\right)\frac{2ik\pi}{z}}\right] = -\frac{\pi}{z}\cdot\frac{z}{ik\pi}\cdot\frac{1}{4}i\sin\left(\sqrt{(k\pi)^2-z^2}-k\pi\right) = -\frac{1}{4k}\sin\sqrt{(k\pi)^2-z^2}\,\cos k\pi =$$

$$= \frac{(-1)^{k+1}}{4k}\sin\sqrt{(k\pi)^2-z^2}$$

$$\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}} J_0\left(z\sqrt{1-a^2\operatorname{ch}^2 t}\right)\operatorname{ch}t\,dt = \left| u = \sqrt{\frac{1-a^2\operatorname{ch}^2 t}{1-a^2}},\ du = \frac{1}{\sqrt{1-a^2}}\left(-\frac{a^2\operatorname{sh}t\operatorname{ch}t\,dt}{\sqrt{1-a^2\operatorname{ch}^2t}}\right) = -\frac{a^2}{u(1-a^2)}\cdot\sqrt{\frac{1-(1-a^2)u^2}{a^2}}\cdot\sqrt{\frac{1-(1-a^2)u^2}{a^2}-1}\,dt = \right.$$

$$\left. = \frac{\sqrt{1-(1-a^2)u^2}\cdot\sqrt{1-a^2}\sqrt{1-u^2}}{(1-a^2)u}\,dt \right| = \int_1^0 J_0\left(z\sqrt{1-a^2}\,u\right)\cdot\frac{\sqrt{1-(1-a^2)u^2}}{a}\left(-\frac{(1-a^2)u}{\sqrt{1-(1-a^2)u}\sqrt{1-a^2}\sqrt{1-u^2}}\right)du =$$

$$= \frac{\sqrt{1-a^2}}{a}\int_0^1 \frac{u\,J_0\left(z\sqrt{1-a^2}\cdot u\right)}{\sqrt{1-u^2}}\,du \;\boxed{=}$$

$$\int_0^1 \frac{x\,J_0(xy)}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{x}{\sqrt{1-x^2}}\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(-\frac{x^2y^2}{4}\right)^n dx = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{2^{2n}(n!)^2}\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx \;=$$

$$\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}dx = \int_0^1\frac{x^{2n-1}(x^2-1+1)}{\sqrt{1-x^2}}dx = -\int_0^1 x^{2n-1}\sqrt{1-x^2}\,dx + \int_0^1\frac{x^{2n-1}}{\sqrt{1-x^2}}dx = -\frac{1}{2n}\left(\left(x^{2n}\sqrt{1-x^2}\right)\Big|_0^1 - \int_0^1 x^{2n}\cdot\left(-\frac{x}{\sqrt{1-x^2}}\right)dx\right) + \int_0^1\frac{x^{2n-1}}{\sqrt{1-x^2}}dx$$

$$= -\frac{1}{2n}\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}dx + \int_0^1\frac{x^{2n-1}}{\sqrt{1-x^2}}dx = \frac{1}{1+\frac{1}{2n}}\int_0^1\frac{x^{2n-1}}{\sqrt{1-x^2}}dx = \frac{2n}{2n+1}\int_0^1\frac{x^{2n-1}}{\sqrt{1-x^2}}dx = \frac{2n(2n-2)\cdot\ldots\cdot 2}{(2n+1)(2n-1)\cdot\ldots\cdot 3}\int_0^1\frac{x\,dx}{\sqrt{1-x^2}} =$$

$$= \frac{(2n)!!}{(2n+1)!!}\cdot\frac{1}{2}\cdot 2\left(-\sqrt{1-x^2}\right)\Big|_0^1 = \frac{(2n)!!}{(2n+1)!!} = \frac{\left((2n)!!\right)^2}{(2n+1)!} = \frac{\left(2^n n!\right)^2}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{2^{2n}(n!)^2}\cdot\frac{\left(2^n\cdot n!\right)^2}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{(2n+1)!} = \frac{\sin y}{y}$$

$$\boxed{=}\ \frac{\sqrt{1-a^2}}{a}\cdot\frac{\sin\left(z\sqrt{1-a^2}\right)}{z\sqrt{1-a^2}} = \frac{\sin\left(z\sqrt{1-a^2}\right)}{az}$$

$$\int_0^{\ln\frac{1+\sqrt{1-a^2}}{a}}\frac{J_1\left(z\sqrt{1-a^2\operatorname{ch}^2 t}\right)}{\sqrt{1-a^2\operatorname{ch}^2t}}\operatorname{ch}t\,dt = \int_0^1\frac{\sqrt{1-a^2}}{a}\cdot\frac{J_1\left(z\sqrt{1-a^2}\cdot u\right)}{\sqrt{1-a^2}\cdot u}\cdot\frac{u}{\sqrt{1-u^2}}\,du = \frac{1}{a}\int_0^1\frac{J_1\left(z\sqrt{1-a^2}\cdot u\right)}{\sqrt{1-u^2}}\,du \;=$$

$$\int_0^1\frac{J_1(xy)}{\sqrt{1-x^2}}dx = \int_0^1\frac{1}{\sqrt{1-x^2}}\cdot\frac{xy}{2}\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\left(-\left(\frac{xy}{2}\right)^2\right)^n dx = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{2^{2n+1}\cdot n!(n+1)!}\int_0^1\frac{x^{2n+1}}{\sqrt{1-x^2}}dx = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{2^{2n+1}(n!)^2(n+1)}\cdot\frac{\left(2^n\cdot n!\right)^2}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{2(n+1)(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{(2n+2)!} = -\frac{1}{y}(\cos y - 1)$$

$$= \frac{1}{a}\left(-\frac{1}{z\sqrt{1-a^2}}\left(\cos\left(z\sqrt{1-a^2}\right)-1\right)\right) = -\frac{\cos\left(z\sqrt{1-a^2}\right)-1}{az\sqrt{1-a^2}}$$

$$\int_0^z \frac{y\,\mathrm{sh}\,y\,\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy = |y=z\cos\theta| = \int_{\pi/2}^0 \frac{z\cos\theta\,\mathrm{sh}(z\cos\theta)\sin(z\sin\theta)}{(z\sin\theta)^2}(-z\sin\theta)\,d\theta = \frac12\int_0^{\pi}\mathrm{sh}(z\cos\theta)\sin(z\sin\theta)\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = |x=e^{i\theta}| =$$

$$= \frac12\int_{\curvearrowleft}\frac12\left(e^{\frac z2(x+\frac1x)}-e^{-\frac z2(x+\frac1x)}\right)\cdot\frac1{2i}\left(e^{\frac z2(x-\frac1x)}-e^{-\frac z2(x-\frac1x)}\right)\cdot\frac{\frac12(x+\frac1x)}{\frac1{2i}(x-\frac1x)}\frac{dx}{ix} = \frac1{8i}\int_{\curvearrowleft}\left(e^{zx}-e^{-\frac zx}-e^{\frac zx}+e^{-zx}\right)\frac{x+\frac1x}{x-\frac1x}\frac{dx}{x} =$$

$$\int_{\curvearrowleft} e^{\frac zx}\frac{x+\frac1x}{x-\frac1x}\frac{dx}{x} = \left|x=\frac1y\right| = \int_{\curvearrowright} e^{zy}\frac{\frac1y+y}{\frac1y-y}\cdot y\cdot\left(-\frac1{y^2}\right)dy = \int_{\curvearrowright} e^{zy}\,\frac{y+\frac1y}{y-\frac1y}\frac{dy}{y}$$

$$= \frac1{8i}\oint_{|x|=1}\left(e^{zx}+e^{-zx}\right)\frac{x+\frac1x}{x-\frac1x}\frac{dx}{x} = \frac1{4i}\oint_{|x|=1}\frac{(x^2+1)\,\mathrm{ch}\,zx}{x(x^2-1)}\,dx =$$

$$= \frac1{4i}\left(2\pi i\cdot(-1)+\pi i\left(\frac{2\,\mathrm{ch}\,z}{2}+\frac{2\,\mathrm{ch}(-z)}{-1\cdot(-2)}\right)\right) = \frac\pi2\mathrm{ch}\,z-\frac\pi2 = \frac\pi2(\mathrm{ch}\,z-1)$$

$$\int_0^z\frac{y^2\,\mathrm{ch}(ay)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy = z^2\int_0^z\frac{\mathrm{ch}(ay)\sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy - \int_0^z\mathrm{ch}(ay)\sin\sqrt{z^2-y^2}\,dy$$

$$\int_0^z\mathrm{ch}(ay)\sin\sqrt{z^2-y^2}\,dy = \int_{\pi/2}^0\mathrm{ch}(az\cos\theta)\sin(z\sin\theta)(-z\sin\theta)\,d\theta = \frac z2\int_0^\pi\mathrm{ch}(az\cos\theta)\sin(z\sin\theta)\sin\theta\,d\theta =$$

$$= \frac z2\int_{\curvearrowleft}\left(e^{\frac{az}2(x+\frac1x)}+e^{-\frac{az}2(x+\frac1x)}\right)\cdot\frac12\cdot\frac1{2i}\left(e^{\frac z2(x-\frac1x)}-e^{-\frac z2(x-\frac1x)}\right)\cdot\frac1{2i}\left(x-\frac1x\right)\frac{dx}{ix} =$$

$$= +\frac{iz}{16}\int_{\curvearrowleft}\left(e^{\frac z2((a+1)x+\frac{a-1}x)}+e^{-\frac z2((a-1)x+\frac{a+1}x)}+e^{\frac z2((a-1)x+\frac{a+1}x)}-e^{-\frac z2((a+1)x+\frac{a-1}x)}\right)\left(x-\frac1x\right)\frac{dx}{ix} =$$

$$= +\frac{iz}{16}\oint_{|x|=1}\left(e^{\frac z2((a+1)+\frac{a-1}x)}-e^{-\frac z2((a+1)x+\frac{a-1}x)}\right)\left(x-\frac1x\right)\frac{dx}{x} = +\frac{iz}{8}\oint_{|x|=1}\frac{(x^2-1)\,\mathrm{sh}\left(\frac z2\left((a+1)x+\frac{a-1}x\right)\right)}{x^2}\,dx =$$

$$e^{\frac z2((a+1)+\frac{a-1}x)} = \sum_{n=0}^\infty\sum_{k=0}^\infty\frac{(a+1)^n(a-1)^k}{n!\,k!}\left(\frac z2\right)^{n+k}x^{n-k}$$

$n-k=-1$, $k=n+1$; $\quad n-k-2=-1$, $n=k+1$

$$\oint_{|x|=1}e^{\frac z2((a+1)x+\frac{a-1}x)}\,dx = 2\pi i\cdot\sum_{n=0}^\infty\frac{(a+1)^n(a-1)^{n+1}}{n!\,(n+1)!}\left(\frac z2\right)^{2n+1} = 2\pi i\cdot\left(-\sqrt{\frac{1-a}{1+a}}\,\frac z2\sqrt{1-a^2}\sum_{n=0}^\infty\frac{\left(-\frac{z^2}{4}\sqrt{1-a^2}\right)^n}{n!\,(n+1)!}\right) = -2\pi i\sqrt{\frac{1-a}{1+a}}\,J_1\left(z\sqrt{1-a^2}\right)$$

$$\oint_{|x|=1}\frac1{x^2}e^{\frac z2((a+1)x+\frac{a-1}x)}\,dx = 2\pi i\cdot\sum_{n=0}^\infty\frac{(a+1)^{n+1}(a-1)^n}{(n+1)!\,n!}\left(\frac z2\right)^{2n+1} = 2\pi i\cdot\sqrt{\frac{1+a}{1-a}}\,\frac z2\sqrt{1-a^2}\sum_{n=0}^\infty\frac{\left(-\frac{z^2}{4}\sqrt{1-a^2}\right)^n}{n!\,(n+1)!} = 2\pi i\sqrt{\frac{1+a}{1-a}}\,J_1\left(z\sqrt{1-a^2}\right)$$

$$= +\frac{iz}{16}\cdot2\pi i\left(-\sqrt{\frac{1-a}{1+a}}J_1\left(z\sqrt{1-a^2}\right)-\sqrt{\frac{1+a}{1-a}}J_1\left(z\sqrt{1-a^2}\right)+\sqrt{\frac{1-a}{1+a}}J_1\left(-z\sqrt{1-a^2}\right)+\sqrt{\frac{1+a}{1-a}}J_1\left(-z\sqrt{1-a^2}\right)\right) =$$

$$= +\frac{\pi z}{4}\left(\sqrt{\frac{1-a}{1+a}}+\sqrt{\frac{1+a}{1-a}}\right)J_1\left(z\sqrt{1-a^2}\right) = +\frac{\pi z\,J_1\left(z\sqrt{1-a^2}\right)}{2\sqrt{1-a^2}}$$

$$\int_0^z \frac{\operatorname{ch} ay \sin\sqrt{z^2-y^2}}{z^2-y^2}\,dy = \left|y = z\cos\theta\right| = \int_{\pi/2}^0 \frac{\operatorname{ch}(az\cos\theta)\sin(z\sin\theta)}{z^2\sin^2\theta}(-z\sin\theta)\,d\theta = \frac{1}{2z}\int_0^{\pi} \frac{\operatorname{ch}(az\cos\theta)\sin(z\sin\theta)}{\sin\theta}\,d\theta =$$

$$= \left|x=e^{i\theta}\right| = \frac{1}{2z}\int_{\curvearrowright} \frac{\operatorname{ch}\left(\frac{az}{2}\left(x+\frac1x\right)\right)\sin\frac{z}{2i}\left(x-\frac1x\right)}{\frac{1}{2i}\left(x-\frac1x\right)}\frac{dx}{ix} = \frac{1}{2iz}\int_{\curvearrowright} \frac{\frac{1}{4i}\left(e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)} \pm e^{\frac z2\left((a-1)x+\frac{a+1}{x}\right)} + e^{\frac z2\left((1-a)x-\frac{1+a}{x}\right)} - e^{-\frac z2\left((1+a)x+\frac{1-a}{x}\right)}\right)}{\frac{1}{2i}\left(x^2-1\right)} =$$

$$\int_{\curvearrowright} \frac{e^{\pm\frac z2\left((a-1)x+\frac{a+1}{x}\right)}}{x^2-1}\,dx = \left|x=\frac1y\right| = \int \frac{e^{\pm\frac z2\left((a+1)y+\frac{a-1}{y}\right)}}{\left(\frac{1}{y^2}-1\right)}\left(-\frac{1}{y^2}\right)dy = \int \frac{e^{\pm\frac z2\left((a+1)y+\frac{a-1}{y}\right)}}{y^2-1}\,dy$$

$$= \frac{1}{4iz}\oint_{|x|=1} \frac{e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)} - e^{-\frac z2\left((a+1)x+\frac{a-1}{x}\right)}}{x^2-1}\,dx = \frac{1}{2iz}\oint_{|x|=1}\frac{\operatorname{sh}\left(\frac z2\left((a+1)x+\frac{a-1}{x}\right)\right)}{x^2-1}\,dx =$$

$$\frac{e^{\frac z2\left((a+1)x+\frac{a-1}{x}\right)}}{x^2-1} = \sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{m=0}^\infty \left(\frac{z(a+1)}{2}x\right)^n\frac{1}{n!}\left(\frac{z(a-1)}{2x}\right)^k\frac{1}{k!}\cdot\left(-x^{2m}\right) = -\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(a+1)^n(a-1)^k}{n!\,k!}\left(\frac z2\right)^{n+k} x^{n-k+2m}$$

exp

$n-k+2m=-1,\ k=n+2m+1$

$$\operatorname*{res}_{x=0}\frac{\exp\left(\frac z2\left((a+1)x+\frac{a-1}{x}\right)\right)}{x^2-1} = -\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(a+1)^n(a-1)^{n+2m+1}}{n!\,(n+2m+1)!}\left(\frac z2\right)^{2n+2m+1} = -\sum_{m=0}^\infty\left(\frac z2(a-1)\right)^{2m+1}\sum_{n=0}^\infty\frac{(a+1)^n\left(\frac z2\right)^{2n}(a-1)^n}{n!\,(n+2m+1)!} =$$

$$= -\sum_{m=0}^\infty (-1)^{2m+1}\left(\frac z2\sqrt{1-a^2}\right)^{2m+1}\cdot\left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}}\cdot\sum_{n=0}^\infty\frac{\left(-\frac z2\sqrt{1-a^2}\right)^{2n}}{n!\,(n+2m+1)!} =$$

$$= \sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(z\sqrt{1-a^2}\right)$$

$$\operatorname*{res}_{x=0}\frac{\operatorname{sh}\left(\frac z2\left((a+1)x+\frac{a-1}{x}\right)\right)}{x^2-1} = \frac12\left(\sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(z\sqrt{1-a^2}\right) - \sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(-z\sqrt{1-a^2}\right)\right) = \sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(z\sqrt{1-a^2}\right)$$

$$\operatorname*{res}_{x=\pm1}\frac{\operatorname{sh}\left(\frac z2\left((a+1)x+\frac{a-1}{x}\right)\right)}{x^2-1} = \frac{e^{\pm az}}{\pm 2}$$

$$= \frac{1}{2iz}\cdot\left(2\pi i\sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(z\sqrt{1-a^2}\right) + \pi i\left(\frac{e^{az}}{2}-\frac{e^{-az}}{2}\right)\right) = \frac{\pi}{2z}\operatorname{sh} az + \frac{\pi}{z}\sum_{m=0}^\infty \left(\frac{1-a}{1+a}\right)^{\frac{2m+1}{2}} J_{2m+1}\left(z\sqrt{1-a^2}\right)$$