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On the derivation of new families of generating functions involving ordinary Bessel functions and Bessel–Hermite functions

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Abstract

It is shown that new families of generating functions involving standard and non-standard forms of Bessel functions can be obtained by means of a fairly simple formalism employing operational methods.

Keywords: Generating functions; Hermite–Kampé de Fériet polynomials; Bessel functions; Tricomi–Bessel functions

1. Introduction

The Bessel functions satisfy a plethora of generating functions which are well documented in the mathematical literature [1], in this paper we consider the derivation of new families of such generating functions, as e.g.

$$G(x; t|m) = \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} J_{mr}(x) \tag{1}$$

where m is an integer and $J_n(x)$ is a cylindrical Bessel function of the first kind.

We will show that their explicit derivation requires the use of Hermite–Bessel functions, i.e. families of functions using Hermite polynomials as the basis for expansion [2].

Before getting into the specific aspects of the paper, we recall some notions relevant to the higher order Hermite–Kampé de Fériet (HKdF) polynomials [3] and to the Tricomi–Bessel functions we will exploit in this paper.

The HKdF polynomials of mth order are defined as (when $m = 2$, the index m is not explicitly reported)

$$H_n^{(m)}(x, y) = n! \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{y^r x^{n-mr}}{(n-mr)!r!} \tag{2}$$

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and satisfy the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x, y) = \exp(xt + yt^m). \tag{3}$$

The Tricomi–Bessel functions read

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \tag{4}$$

and the relevant generating functions writes

$$\sum_{n=-\infty}^{\infty} t^n C_n(x) = \exp\left(t - \frac{x}{t}\right). \tag{5}$$

The HKdF polynomials can also be defined by exploiting the operational rule [2]

$$\exp\left(y \frac{\partial^m}{\partial x^m}\right) x^n = H_n^{(m)}(x, y). \tag{6}$$

By noting that the Hermite–Tricomi functions are defined through the identity

$$C_n^{(m)}(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r^{(m)}(x, y)}{r!(n+r)!} \tag{7}$$

we find, on account of Eqs. (5) and (6) that

$$\sum_{n=-\infty}^{\infty} t^n C_n^{(m)}(x, y) = \exp\left(t - \frac{x}{t} + \frac{y}{(-t)^m}\right) \tag{8}$$

$$\exp\left(y \frac{\partial^m}{\partial x^m}\right) C_n(x) = C_n^{(m)}(x, y).$$

In the following sections we will see how the above results can be exploited to obtain the generating function given in Eq. (1).

2. Bessel generating functions

The Tricomi functions are specified by the following further property

$$(-1)^s \left(\frac{\partial}{\partial x}\right)^s C_0(x) = C_s(x) \tag{9}$$

and are linked to the ordinary cylindrical functions by

$$J_n(x) = \left(\frac{x}{2}\right)^n C_n\left(\frac{x^2}{4}\right). \tag{10}$$

On account of Eq. (1), we get

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{mr}(x) = \exp\left((-1)^{m+1} t \frac{\partial^m}{\partial x^m}\right) C_0(x) \tag{11}$$

which, according to the second of Eq. (8), can be written in terms of HKdF Tricomi functions as follows

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{mr}(x) = C_0^{(m)}\left(x, (-1)^{m+1} t\right). \tag{12}$$

The case of the ordinary cylindrical functions can be treated in the same way and indeed, owing to Eq. (10), we find

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} J_{mr}(x) = \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \left(\frac{x}{2}\right)^{mr} C_{mr}\left(\frac{x^2}{4}\right) \tag{13}$$

and, finally, according to Eq. (12) we get

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} J_{mr}(x) = C_0^{(m)}\left(\frac{x^2}{4}, (-1)^{m+1}\frac{x^m t}{2^m}\right). \tag{14}$$

The previous results have shown the usefulness of the Tricomi functions, and of the associated HKdF forms, within the context of Bessel generating function problems.

We will now prove that further progress can be made in the derivation of further generating functions associated with Bessel type functions, by means of an extension of the above outlined formalism. To this end, we consider the case

$$S(x; t|m) = \sum_{r=0}^{\infty} (-t)^r C_{mr}(x), \quad |t| < 1, \tag{15}$$

and the use of the second of Eq. (8) yields the operational identity

$$\sum_{r=0}^{\infty} (-t)^r C_{mr}(x) = \frac{1}{1-(-1)^{m+1} t \frac{\partial}{\partial x}} C_0(x) \tag{16}$$

which, according to elementary Laplace transform methods, can be recast in a more convenient integral form, namely

$$\frac{1}{1-(-1)^{m+1} t \frac{\partial}{\partial x}} C_0(x) = \int_0^{\infty} \exp(-s) \exp\left((-1)^{m+1} s t \frac{\partial}{\partial x}\right) C_0(x) \mathrm{d}s. \tag{17}$$

According to the previous discussion, it is therefore quite straightforward to end up with

$$\sum_{r=0}^{\infty} (-t)^r C_{mr}(x) = \int_0^{\infty} \mathrm{d}s \exp(-s) C_0^1\left(x, (-1)^{m+1} s t\right). \tag{18}$$

We can eliminate the integral by noting that

$$\int_0^{\infty} \mathrm{d}s \exp(-s) H_n^{(m)}(x, ys) = \bar{e}_n^{(m)}(x, y) \tag{19}$$

where

$$\bar{e}_n^{(m)}(x, y) = n! \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{x^{n-mr} y^r}{(n-mr)!} \tag{20}$$

are truncated exponential forms, of the type discussed in Ref. [4]. They satisfy the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \bar{e}_n^{(m)}(x, y) = \frac{\exp(xt)}{1-t^m}. \tag{21}$$

These polynomials, too, can be exploited to define new families of Bessel functions, so that we find

$$\sum_{r=0}^{\infty} (-t)^r C_{mr}(x) = {}_e C_0^{(1)}(x, (-1)^{m+1} t), \tag{22}$$

$${}_e C_n^{(m)}(x, y) = \sum_{r=0}^{\infty} \frac{(-1)^r \bar{e}_r^{(m)}(x, y)}{r!(n+r)!}.$$

The results of this paper are further proof of the effectiveness of the success of operational methods once applied to the derivation of generating functions; further examples will be discussed in the forthcoming concluding remarks.

3. Concluding remarks

In the previous section we have mainly dealt with problems involving just a single Bessel function; here we will show that the method can be easily extended to more general cases, involving e.g. products of Bessel functions and special polynomials.

Before getting into this specific aspect of the problem, let us recall that we can define multi-dimensional Hermite polynomials, using a generalization of the operational method discussed in the introductory remarks. We consider indeed the case [2]

$$H_n^{(3)}(x, y, z) = n! \sum_{r=0}^{[\frac{n}{3}]} \frac{z^r H_{n-3r}(x, y)}{(n-3r)!r!} \tag{23}$$

which can be defined according to the operational rule

$$\begin{aligned} H_n^{(3)}(x, y, z) &= \exp\left(z \frac{\partial^3}{\partial x^3}\right) H_n(x, y) \\ &= \exp\left(z \frac{\partial^3}{\partial x^3} + y \frac{\partial^2}{\partial x^2}\right) x^n. \end{aligned} \tag{24}$$

More generally, we find ($\{x\}_1^m = x_1, \ldots, x_m$)

$$H_n^{(m)}(\{x\}_1^m) = \exp\left(\sum_{s=2}^{m} x_s \left(\frac{\partial}{\partial x_1}\right)^s\right) x_1^n. \tag{25}$$

In the context of these polynomials we can introduce the Tricomi–Bessel defined as

$$C_n^{(m)}(\{x\}_1^m) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r^{(m)}(\{x\}_1^m)}{r!(n+r)!}. \tag{26}$$

It is also evident that an operational definition analogous to (25) can be exploited to introduce the Tricomi–Bessel, namely

$$C_n^{(m)}(\{x\}_1^m) = \exp\left(\sum_{s=2}^{m} x_s \left(\frac{\partial}{\partial x_1}\right)^s\right) C_n(x_1).$$

We can now apply the previous considerations to derive generating functions of the type

$$\Phi(x, y, z; t|m) = \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{mr}(x) H_r(z, y), \tag{27}$$

which can be recast in the form

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{mr}(x) H_r(z, y) = \exp\left(y \frac{\partial^2}{\partial z^2}\right) C_0^{(m)}(x, (-1)^{m+1} tz). \tag{28}$$

The problem is now the evaluation of the action of the previous exponential polynomial on the Tricomi–Bessel function.

We consider the case with $m = 2$ first. By noting that [2]

$$\exp\left(y \frac{\partial^2}{\partial z^2}\right) H_n(x, az) = H_n^{(4)}(x, az, 0, a^2 y) \tag{29}$$

we find

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{2r}(x) H_r(z, y) = C_0^{(4)}(x, -tz, 0, t^2 y) \tag{30}$$

and in an analogous way we also get

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} J_{2r}(x) H_r(z, y) = C_0 \left(\frac{x^2}{4}, -\frac{x^2}{4} tz, 0, y \frac{x^2}{4} t^2 \right). \tag{31}$$

The case with $m \geq 2$ requires more calculations. We first note, therefore, that

$$\exp\left(y \frac{\partial^2}{\partial z^2} \right) H_n^{(m)}(x, az) = H_n^{(2m,m)}(x, az, a^2 z), \tag{32}$$

where we have defined

$$H_n^{(p,m)}(x, y, z) = n! \sum_{r=0}^{\left[\frac{n}{p}\right]} \frac{z^r H_{n-pr}^{(m)}(x, y)}{(n - pr)! r!} \tag{33}$$

so that we get

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} C_{mr}(x) H_r(z, y) = C_0^{(2m,m)}(x, (-1)^{m+1} tz, t^2 y), \tag{34}$$

where we have defined

$$C_n^{(m,p)}(x, y, z) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r^{(m,p)}(x, y, z)}{r!(n + r)!}. \tag{35}$$

The extension to ordinary Bessel functions does not present any problem and reads

$$\sum_{r=0}^{\infty} \frac{(-t)^r}{r!} J_{mr}(x) H_r(z, y) = C_0^{(2m,m)} \left(\frac{x^2}{4}, (-1)^{m+1} \frac{x^{2m}}{2^{2m}} tz, t^2 \frac{x^{2m}}{2^{2m}} y \right). \tag{36}$$

In a forthcoming investigation we will investigate how the above results can be further extended.

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